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Linear problems of the stability of a type of rotation of a satellite about the centre of mass[☆]

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ABSTRACT

The stability in the first approximation of the rotation of a satellite about a centre of mass is investigated. In the unperturbed motion the satellite performs, in absolute space, three rotations around the normal to the orbital plane in a time equal to two periods of rotation of its centre of mass in the orbit (Mercury-type rotation). Three cases of such rotations are considered: the rotations of a dynamically symmetrical satellite and a satellite, the central ellipsoid of inertia of which is close to a sphere, in an elliptic orbit of arbitrary eccentricity, and the rotation of a satellite with three different principal central moments of inertia in a circular orbit.

1. The Hamiltonian function

Consider the motion of a satellite–rigid body about a centre of mass under the action of the gravitational moments of a central Newtonian force field. The linear dimensions of the satellite are assumed to be small compared with the characteristic size of the orbit, which enables us to assume,[1] that its motion about the centre of mass has no effect on the motion of the centre of mass itself. It is assumed that the centre of mass of the satellite moves in an elliptical orbit with eccentricity $e(0 \le e < 1)$.

Suppose *OXYZ* is an orbital system of coordinates with origin at the centre of mass *O* of the satellite. Its *OZ* axis is directed along the radius vector of the centre of mass relative to an attracting centre, while the *OX* and *OY* axes are parallel to the transversal and normal to the plane of the orbit respectively. In absolute space, the trihedron *OXYZ* rotates around the *OY* axis with angular velocity

$$\omega = \frac{d\nu}{dt} = \frac{\omega_0}{(1-e^2)^{3/2}}\zeta^2; \quad \omega_0 = \frac{2\pi}{\tau}, \quad \zeta = 1 + e\cos\nu \tag{1.1}$$

where ν is the true anomaly and τ is the period of rotation of the centre of mass in the orbit.

The system of coordinates *Oxyz* is rigidly connected to the satellite and its axes are directed along the principal central axes of inertia of the satellite. We will denote the moments of inertia corresponding to these axes by *A*, *B* and *C*. The mutual orientation of the trihedrons *Oxyz* and *OXYZ* will be specified using the Euler angles ψ, θ, φ.

The kinetic energy of the satellite in its motion about the centre of mass is calculated from the formula

$$T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) \tag{1.2}$$

where *p*, *q* and *r* are the projections of the absolute angular velocity of the satellite onto the *Ox*, *Oy* and *Oz* axes:

$$\begin{aligned}
&p = \dot\psi\sin\varphi\sin\theta + \dot\theta\cos\varphi + \omega a_{21}, \quad q = \dot\psi\cos\varphi\sin\theta - \dot\theta\sin\varphi + \omega a_{22}\\
&r = \dot\psi\cos\theta + \dot\varphi - \omega\cos\psi\sin\theta\\
&a_{21} = \sin\psi\cos\varphi + \cos\psi\sin\varphi\cos\theta, \quad a_{22} = -\sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta
\end{aligned} \tag{1.3}$$

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The potential energy of the satellite in the problem of its motion about the centre of mass has the form[1]

$$\Pi = \frac{3}{2}\frac{\omega_0^2}{(1-e^2)^3}\zeta^3[(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + C\cos^2\theta] \tag{1.4}$$

Using the Lagrange function $L = T - \Pi$ we introduce the generalized momenta

$$P_\psi = \frac{\partial L}{\partial\dot\psi},\quad P_\theta = \frac{\partial L}{\partial\dot\theta},\quad P_\varphi = \frac{\partial L}{\partial\dot\varphi} \tag{1.5}$$

If T_2 is the part of the kinetic energy which is a quadratic form with respect to $\dot\psi$, $\dot\theta$, $\dot\varphi$ while T_0 is the part which is independent of $\dot\psi$, $\dot\theta$, $\dot\varphi$, the Hamiltonian function can be calculated from the formula[2]

$$H = T_2 - T_0 + \Pi \tag{1.6}$$

on the right-hand side of which the quantities $\dot\psi$, $\dot\theta$, $\dot\varphi$ are replaced by their expressions in terms of ψ, θ, φ, P_ψ, P_θ, P_φ obtained from relations (1.5).

If, using the equalities

$$P_\psi = \tilde{A}p_\psi,\quad P_\theta = \tilde{A}p_\theta,\quad P_\varphi = \tilde{A}p_\varphi;\quad \tilde{A} = A\omega_0(1-e^2)^{-3/2} \tag{1.7}$$

we introduce the dimensionless momenta p_ψ, p_θ, p_φ and, using Eq. (1.1), we change to a new independent variable, namely, the true anomaly, then the expression for the Hamiltonian function (1.6) can be written in the form

$$\begin{aligned} H = {} & \frac{A\cos^2\varphi + B\sin^2\varphi}{2B\zeta^2}\left(\frac{p_\psi - p_\varphi\cos\theta}{\sin\theta}\right)^2 + \frac{A\sin^2\varphi + B\cos^2\varphi}{2B\zeta^2}p_\theta^2 + \\ & + \frac{A}{2C\zeta^2}p_\varphi^2 + \frac{(B-A)\sin 2\varphi}{2B\zeta^2}\left(\frac{p_\psi - p_\varphi\cos\theta}{\sin\theta}\right)p_\theta - \left(\frac{p_\psi\cos\theta - p_\varphi}{\sin\theta}\right)\cos\psi - \\ & - p_\theta\sin\psi + \frac{3}{2}\zeta\left[\frac{B-A}{A}\cos^2\varphi\sin^2\theta + \frac{C-A}{A}\cos^2\theta\right] \end{aligned} \tag{1.8}$$

2. The stability of the rotation of a dynamically symmetrical satellite in an elliptic orbit

If the satellite is dynamically symmetrical ($A = B$), then φ is a cyclic coordinate and

$$P_\varphi = \tilde{A}p_\varphi = Cr_0 = \text{const} \tag{2.1}$$

$$\dot\psi\cos\theta + \dot\varphi - \omega\cos\psi\sin\theta = r_0 \tag{2.2}$$

Here r_0 is the projection of the absolute angular velocity of the satellite onto its axis of symmetry Oz, which, when $A = B$, is constant.

Substituting the expression for p_φ, obtained from (2.1), into the function (1.8) and putting $A = B$ in it, we arrive[3] at a reduced system with two degrees of freedom with Hamiltonian function $H = H(\psi, \theta, p_\psi, p_\theta, \nu; e, \alpha, \beta)$. The reduced system describes the motion of the axis of symmetry of the satellite in an orbital system of coordinates. If this motion is obtained, the rotation of the satellite around the axis of symmetry (i.e. the relation $\varphi = \varphi(\nu)$) is found from integral (2.2) using a quadrature. The Hamiltonian function of the reduced system, in addition to the eccentricity of the orbit e, depends on two more dimensionless parameters

$$\alpha = C/A,\quad \beta = r_0/\omega_0\ (0 < \alpha \le 2, -\infty < \beta < \infty)$$

Steady rotation and the Hamiltonian function of perturbed motion. The reduced system, for any physically allowable values of the parameters α, β and e, allows of the particular solution[4]

$$\theta = \pi/2,\quad \psi = \pi,\quad p_\theta = 0,\quad p_\psi = 0 \tag{2.3}$$

in which the axis of symmetry of the satellite Oz is perpendicular to the orbital plane, while the satellite itself rotates around this axis with constant angular velocity r_0.

In the Hamiltonian function $H = H(\psi, \theta, p_\psi, p_\theta, \nu; e, \alpha, \beta)$ of the reduced system we put

$$\theta = \pi/2 + q_1,\quad \psi = \pi + q_2,\quad p_\theta = p_1,\quad p_\psi = p_2$$

and expand it in series in powers of q_i, p_i ($i=1,2$). The corresponding quadratic part of the Hamiltonian function of the perturbed motion of the linear problem of the stability of steady rotation (2.3) has the form

$$H_2 = \frac{1}{2}\left[\frac{\eta^2}{\zeta^2} - \eta + 3(\alpha-1)\zeta\right]q_1^2 + \left(\frac{\eta}{\zeta^2} - 1\right)q_1 p_2 + \frac{1}{2}\eta q_2^2 +$$
$$+ q_2 p_1 + \frac{1}{2\zeta^2}(p_1^2 + p_2^2); \quad \eta = \alpha\beta(1-e^2)^{3/2} \tag{2.4}$$

For Mercury-type steady rotation, the value of r_0 is equal to $3\omega_0/2$ (i.e. $\beta = 3/2$).

Regions of stability and instability when $\beta = 3/2$. Suppose we put $\beta = 3/2$ in the Hamiltonian function (2.4). Again putting $e = 0$, we obtain the Hamiltonian function corresponding to the linearized equations of perturbed motion of the satellite in the neighbourhood of its Mercury-type rotation in a circular orbit. When $0 < \alpha < 8/9$, $\alpha \neq 2/3$ the rotation considered is unstable in a circular orbit, since there is a root with a positive real part in the characteristic equation.[1,5] When $\alpha = \alpha_0 = 2/3$ and $\alpha = \alpha_* = 8/9$ there are no roots with positive real part in the characteristic equation, but there are multiple zero roots; in these two cases there is also instability in the first approximation.[6] For the remaining physically allowed values of the parameter α,

$$\alpha_* < \alpha \le 2 \tag{2.5}$$

the rotation of the satellite is stable.[7] For values of α from the range (2.5) the Hamiltonian function (2.4) (when $\beta = 3/2$ and $e = 0$) is positive definite, while the roots of the characteristic equation $\pm i\omega_1$, $\pm i\omega_2$ are pure imaginary and different, where the frequencies ω_1 and $\omega_2 (\omega_1 > \omega_2 > 0)$ of small oscillations of the axis of symmetry of the satellite in the neighbourhood of the normal to the orbital plane are the roots of the equation

$$4\omega^4 - (9\alpha^2 - 4)\omega^2 + (3\alpha-2)(9\alpha-8) = 0 \tag{2.6}$$

For small but non-zero values of the eccentricity e, the regions of stability and instability can be obtained, for example, using the Deprit–Hori method of canonical replacements of variables.[8,9] Bearing in mind the sign-definiteness of the function (2.4) when $e = 0$, using the Krein–Gel'fand–Lidskii theorem,[10] we obtain that, in the range (2.5), the following seven points are generating points for regions of parametric resonance (the corresponding resonance relations for the frequencies ω_1 and ω_2 are given in brackets):

$$\alpha_1 = 0.895178(\omega_1 + \omega_2 = 1), \quad \alpha_2 = 1.166761(\omega_1 + \omega_2 = 2)$$
$$\alpha_3 = 4/3(\omega_2 = 1), \quad \alpha_4 = 1.386203(2\omega_1 = 3), \quad \alpha_5 = 1.600546(\omega_1 + \omega_2 = 3)$$
$$\alpha_6 = 1.680532(\omega_1 = 2), \quad \alpha_7 = 1.976312(2\omega_1 = 5) \tag{2.7}$$

The boundaries of the regions of instability, emerging from these points when $e = 0$, for small e are given by the equations

$$\alpha_1^{\pm} = \alpha_1 \pm 0.02046e - 0.0739e^2 \pm 0.195e^3 + O(e^4)$$
$$\alpha_2^{+} = \alpha_2 + 0.2926e^2 + O(e^4), \quad \alpha_2^{-} = \alpha_2 - 0.3879e^2 + O(e^4)$$
$$\alpha_3^{+} = 4/3 - e^2 + O(e^4), \quad \alpha_3^{-} = 4/3 - (5/3)e^2 + O(e^4)$$
$$\alpha_4^{\pm} = \alpha_4 + 0.6156e^2 \pm 1.247e^3 + O(e^4)$$
$$\alpha_5^{\pm} = \alpha_5 - 0.1874e^2 \pm 0.499e^3 + O(e^4)$$
$$\alpha_6^{+} = \alpha_6 + 3.2668e^2 - 14.74e^4 + O(e^5)$$
$$\alpha_6^{-} = \alpha_6 + 3.2668e^2 - 22.22e^4 + O(e^5)$$
$$\alpha_7^{\pm} = \alpha_7 - 2.7957e^2 + 15.45e^4 \pm 0.7e^5 + O(e^6)$$

The boundary of the region of instability, issuing from the point $\alpha_* = 8/9$ for small e, is described by the equation

$$\alpha_*(e) = \frac{8}{9} - \frac{4}{27}e^2 + O(e^4)$$

The regions of stability and instability in the neighbourhood of the point $\alpha = \alpha_0 = 2/3$ were obtained previously in Ref. 9. In the case considered here, $\beta = 3/2$ and, for small e, the boundaries of the region of stability are given by the equations

$$\alpha_0^{+} = \frac{2}{3} + \frac{9}{64}e^4 + O(e^6), \quad \alpha_0^{-} = \frac{2}{3} - \frac{855}{64}e^4 + O(e^6)$$

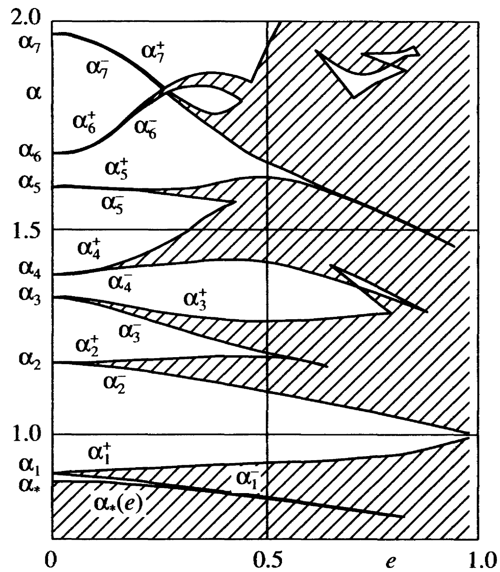

Fig. 1.

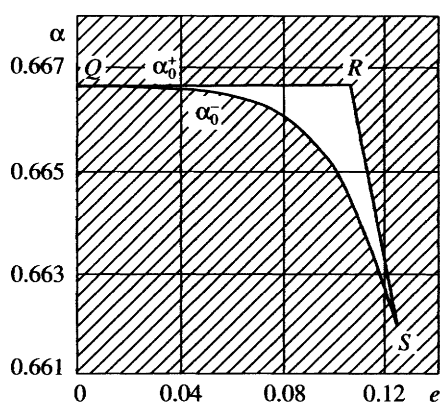

Fig. 2.

For values of e that are not small, the stability was investigated using well-known algorithms[8] by a numerical analysis of the characteristic equation of a linear system, 2π-periodic in ν with Hamiltonian function (2.4) (for $\beta = 3/2$). The calculations were carried out for values of $e < 4.96$.

The results of numerical and analytical investigations of the stability of the steady rotation of a satellite for $\beta = 3/2$ are shown in Figs. 1 and 2 in the plane of the parameters e, α. The regions of instability are shown hatched.

Below the point (0.8200, 0.7988), at which the boundary curve α_1^- and $\alpha_*(e)$ intersect (see Fig. 1), the steady rotation of the satellite is unstable for all e and α, apart from their values in a very narrow region of stability, emerging from the point (0, 2/3). This region is a curvilinear triangle with vertices Q(0, 2/3), R(0.106, 0.667), S(0.125, 0.662) and is shown in Fig. 2.

When $e = 0.2056$ (the eccentricity of the Mercury orbit) there are eight intervals of stability:

$$\begin{aligned}
&0.882632 < \alpha < 0.886648, \quad 0.898842 < \alpha < 1.151760\\
&1.177727 < \alpha < 1.282451, \quad 1.304066 < \alpha < 1.403192\\
&1.425487 < \alpha < 1.592120, \quad 1.594565 < \alpha < 0.787030\\
&1.795246 < \alpha < 1.879065, \quad 1.879706 < \alpha \leq 2
\end{aligned} \tag{2.8}$$

If the value of α does not lie in the intervals (2.8) or on their boundaries, then when $e = 0.2056$, steady rotation of a Mercury-type satellite is unstable.

3. The rotation stability of a satellite when the value of β is close to 3/2

Suppose, as before, that the satellite is dynamically symmetrical and its centre of mass moves in an elliptic orbit. The problem of the stability of steady rotation when it is not Mercury-type rotation but close to it ($0 < |\beta - 3/2| \ll 1$) is of interest. We will consider this problem for a satellite, the central ellipsoid of inertia of which differs only slightly from a sphere ($0 < |\alpha - 1| \ll 1$).

The limiting case $\alpha = 1$, $\beta = 3/2$. We will denote the Hamiltonian function (2.4), calculated for $\alpha = 1$ and $\beta = 3/2$, by $H_2^{(0)}$. Suppose $\mathbf{X}(\nu)$ is the fundamental matrix of the solutions of the linear system of differential equations with Hamiltonian function $H_2^{(0)}$, which satisfies the condition $\mathbf{X}(0) = \mathbf{E}$, where $\mathbf{E}$ is the fourth-order unit matrix. We can obtain the following explicit expression for this

$$\mathbf{X}(\nu) = \begin{Vmatrix} \cos\mu & \sin\nu & f_1(\nu) & -f_2(\nu) \\ -\sin\mu & \cos\nu & f_2(\nu) & f_1(\nu) \\ \kappa\sin\mu & 0 & \cos\mu & \sin\mu \\ 0 & -\kappa\sin\nu & -\sin\nu & \cos\nu \end{Vmatrix}$$

$$\mu = \nu - \frac{3}{2}M, \quad f_1 = \frac{\sin\nu - \sin\mu}{\kappa}, \quad f_2 = \frac{\cos\nu - \cos\mu}{\kappa}, \quad \kappa = \frac{3}{2}(1-e^2)^{3/2}$$

$$M = \int_0^\nu \frac{(1-e^2)^{3/2}}{(1+e\cos x)^2}dx \tag{3.1}$$

($M = M(\nu)$ is the mean anomaly).

The characteristic equation of the matrix $\mathbf{X}(2\pi)$ is independent of the eccentricity and can be written in the form $(\rho^2 - 1)^2 = 0$, where two linearly independent eigenvectors of the matrix $\mathbf{X}(2\pi)$ correspond to each double root of this equation. Hence[11] when $\alpha = 1$ and $\beta = 3/2$ the steady rotation of the satellite (2.3) is stable. This conclusion agrees with the results of numerical calculations (Fig. 1).

Using the canonical univalent replacement of variables, 2π-periodic in ν,

$$\mathbf{u} = \mathbf{N}\boldsymbol{\upsilon}, \quad \mathbf{u}' = (q_1, q_2, p_1, p_2), \quad \boldsymbol{\upsilon}' = (\xi_1, \xi_2, \eta_1, \eta_2) \tag{3.2}$$

the "unperturbed" Hamiltonian function $H_2^{(0)}$ can be reduced to its normal form. According to a well-known algorithm (Ref. 8), the matrix $\mathbf{N}$ can be obtained in the form of a product of three matrices

$$\mathbf{N} = \mathbf{X}(\nu)\mathbf{P}\mathbf{Q}(\nu) \tag{3.3}$$

where

$$\mathbf{P} = \begin{Vmatrix} 0 & 1 & -\kappa^{-1} & 0 \\ 1 & 0 & 0 & -\kappa^{-1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}, \quad \mathbf{Q}(\nu) = \begin{Vmatrix} \cos\nu & 0 & -\sin\nu & 0 \\ 0 & \cos\frac{\nu}{2} & 0 & -\sin\frac{\nu}{2} \\ \sin\nu & 0 & \cos\nu & 0 \\ 0 & \sin\frac{\nu}{2} & 0 & \cos\frac{\nu}{2} \end{Vmatrix} \tag{3.4}$$

while the normal form has the form

$$\Gamma_2^{(0)} = \frac{1}{2}(\xi_1^2 + \eta_1^2) + \frac{1}{4}(\xi_2^2 + \eta_2^2) \tag{3.5}$$

Regions of stability and instability when $\alpha \approx 1$ and $\beta \approx 3/2$. To investigate the stability for small values of the quantities $|\alpha - 1|$ and $|\beta - 3/2|$ we put

$$\alpha = 1 + \varepsilon, \quad \beta = 3/2 + \varepsilon\delta^{(1)} + \ldots \tag{3.6}$$

We substitute expressions (3.6) into function (2.4), expand it in series in powers of ε and then make the replacement of variables using formulae (3.2)–(3.4). As a result we obtain the Hamiltonian function of the perturbed motion in the form of the following series

$$\Gamma_2 = \Gamma_2^{(0)} + \varepsilon\Gamma_2^{(1)} + \ldots \tag{3.7}$$

The function $\Gamma_2^{(0)}$ is defined by (3.5), but the functions $\Gamma_2^{(i)} (i \geq 1)$ are not written explicitly because of their length. There are quadratic forms in $\xi_1, \xi_2, \eta_1, \eta_2$ with coefficients that are 2π-periodic in ν which depend on the excentricity e and the quantities $\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(i)}$ from representation (3.6) of the parameter β in the form of a series. On the boundaries of the regions of stability and instability these quantities are functions of e.

The problem in question contains a multiple (double) parametric resonance, since the frequencies ω_1 and ω_2 of small oscillations of the limit system (when $\varepsilon = 0$) with Hamiltonian function $\Gamma_2^{(0)}$ at once satisfy two resonance relations: $\omega_1 = 1$ and $2\omega_2 = 1$.

Using the algorithm proposed earlier in Ref. 9, we obtain that, in the first approximation in ε, the region of instability is given by the inequality

$$\delta_1^{(1)} < \delta^{(1)} < \delta_2^{(1)} \tag{3.8}$$

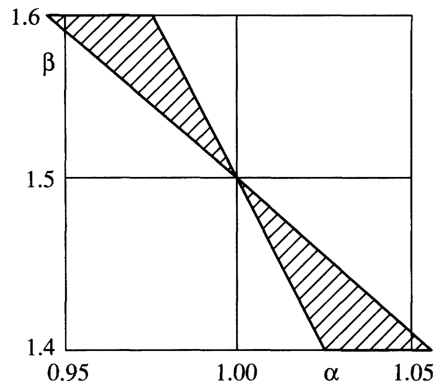

Fig. 3.

where $\delta_j^{(1)}(j=1,2)$ are functions of e. In particular, when $e = 0.2056$ we have

$$\delta_1^{(1)} = -3.2211, \quad \delta_2^{(1)} = -1.9128$$

For small e the functions $\delta_j^{(1)}$ are represented by series of the form

$$\delta_1^{(1)}(e) = -\frac{5}{2} - \frac{7}{2}e - \frac{3}{2}e^2 + \frac{123}{16}e^3 - \frac{15}{8}e^4 - \frac{489}{128}e^5 - \frac{35}{16}e^6 + \ldots, \quad \delta_2^{(1)}(e) = \delta_1^{(1)}(-e)$$

In Fig. 3 we show regions of stability and instability in the neighbourhood of the point $\alpha = 1$, $\beta = 3/2$ for $e = 0.2056$ (the regions of instability are shown hatched). These regions are obtained by analysing the characteristic equation with Hamiltonian function (2.4), obtained by numerical integration. In the range of variation of the parameters α and β, shown in Fig. 3, the boundaries of the regions of stability and instability, obtained by numerical integration, are practically indistinguishable from the approximate boundaries defined by relations (3.8).

If the satellite is dynamically oblate ($\varepsilon > 0$), then, in the region of instability, the value of β should be less than its value corresponding to Mercury-type rotation (when $\beta = 3/2$). For a dynamically prolate satellite ($\varepsilon < 0$), conversely, in the region of instability $\beta > 3/2$.

4. The stability of the rotation of an asymmetrical satellite in a circular orbit

Suppose the satellite is not dynamically symmetrical, while the orbit of its centre of mass is circular ($e = 0$). We will specify the geometry of the mass of the satellite using the following two dimensionless inertial parameters

$$\alpha = C/A, \quad \gamma = 3(A - B)/C$$

We will make the canonical (with valence $4/\alpha$) replacement of variables

$$\theta = \pi/2 + q_1/2, \ \ \psi = \pi + q_2/2, \ \ \varphi = q/2, \ \ p_\theta = \alpha p_1/2, \ \ p_\psi = \alpha p_2/2, \ \ p_\varphi = \alpha(1 + p/2) \tag{4.1}$$

From expression (1.8) and replacements (4.1) we obtain that the Hamiltonian function F, represented in the form of the following series

$$F = F_0 + F_2 + \ldots \tag{4.2}$$

where

$$F_0 = \frac{1}{2}p^2 - \gamma\cos q \tag{4.3}$$

$$F_2 = \frac{1}{4}[f^-(p+2)^2 - p + g + 4]q_1^2 + h(p+2)q_1p_1 +$$

$$+ [f^-(p+2) - 1]q_1p_2 + \frac{1}{4}(p+2)q_2^2 + q_2p_1 + f^+p_1^2 + 2hp_1p_2 + f^-p_2^2$$

$$f^{\mp} = \alpha\frac{\alpha\gamma - 6 \mp \alpha\gamma\cos q}{4(\alpha\gamma - 3)}, \quad g = \frac{\alpha\gamma - 6 + \alpha\gamma\cos q}{\alpha}, \quad h = \frac{\alpha^2\gamma\sin q}{4(\alpha\gamma - 3)} \tag{4.4}$$

corresponds to the motions of a satellite, close to its plane motions about the centre of mass in a circular orbit.

The dots in expansion (4.2) denote the set of forms F_m ($m \geq 4$) of even powers in q_i and p_i ($i = 1,2$). The mean anomaly M is the independent variable.

The unperturbed Mercury-type plane rotation is given by the equalities $q_1 = q_2 = p_1 = p_2 = 0$ and

$$q = (1 - \mathrm{sign}\gamma)\pi/2 + 2\mathrm{am}(\pi^{-1}\mathbf{K}(k)M), \quad p = 2\pi^{-1}\mathbf{K}(k)\mathrm{dn}(\pi^{-1}\mathbf{K}(k)M) \tag{4.5}$$

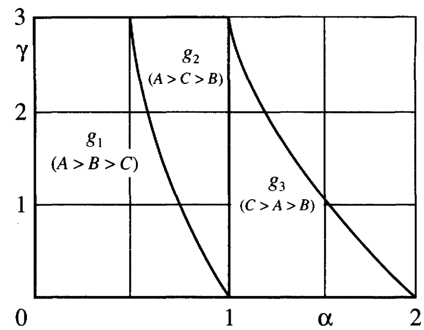

Fig. 4.

Here we have used the generally accepted notation from the theory of elliptic functions and integrals,[12] where the modulus k of the elliptic functions and the parameter γ are connected by the relation

$$|\gamma| = \pi^{-2}k^2\mathbf{K}^2(k) \tag{4.6}$$

Since $0 < |\gamma| \leq 3$, we have $0 < k \leq k_* = 0.99985$.

To solve (4.5) and (4.6) irrespective of the sign of γ at the instant when $M = 0$ and at instants when the mean anomaly M is a multiple of 2π, the axis of the last of the moments of inertia (A or B) is directed along the radius vector of the centre of mass of the satellite.

The function (4.4), in which the quantities q and p are calculated from formulae (4.5) and (4.6), describes the linear problem of the stability of the plane motion considered with respect to spatial perturbations. This function is 2π-periodic in the independent variable M.

Note the following symmetry property of the Hamiltonian function of the perturbed motion. The function (4.4), calculated (taking formulae (4.5) into account) for $\gamma > 0$, by making the replacements

$$\gamma \to -\gamma, \quad \alpha \to 3\alpha/(3 - \alpha\gamma)$$

converts into the same function (4.4), but calculated for $\gamma < 0$. Hence, when constructing the regions of stability and instability we can confine ourselves solely to the case when $\gamma > 0$. The corresponding regions for $\gamma < 0$ are then simply obtained on the basis of this symmetry property.

The range of values of the parameters α and γ considered further is a trapezium-shaped tetragon with vertices (0, 0) (0, 3), (1, 3) and (2, 0), shown in Fig. 4. The side of the tetragon, connecting the vertices (1.3) and (2.0) is part of the hyperbola $\gamma = -3 + 6/\alpha$ ($C = A + B$), while the remaining sides are sections of straight lines. The tetragon is split by the vertical straight line $\alpha = 1$ ($C = A$) and the hyperbola $\gamma = -3 + 3/\alpha$ ($C = B$) into three regions g_1, g_2 and g_3, for which the Oz axis (perpendicular to the orbital plane in unperturbed motion), is the axis of the least, mean and greatest value of the moments of inertia A, B and C respectively.

For $\gamma = 0$ (the satellite is dynamically symmetrical, $A = B$), rotation (4.5), (4.6) changes into the steady rotation of the satellite considered in Section 2 (for $e = 0$). Hence, the conclusions regarding the stability of the rotation (4.5), (4.6) for $\gamma = 0$ are identical with the conclusions in Section 2 regarding the stability of Mercury-type rotation in a circular orbit. The points of the axis α, from which the region of instability originates (i.e. the points $\alpha_0 = 2/3$, $\alpha_* = 8/9$ and the points $\alpha_1, \alpha_2, \ldots, \alpha_7$, given by Eq. (2.7)) will be the same as in Section 2.

We will briefly describe the results of analytical and numerical investigations of the stability of rotation (4.5), (4.6) when $\gamma \neq 0$. In the region g_1, corresponding to rotation around the axis of least moment of inertia, there are three regions of stability 1, 2 and 3 (Fig. 5). Region

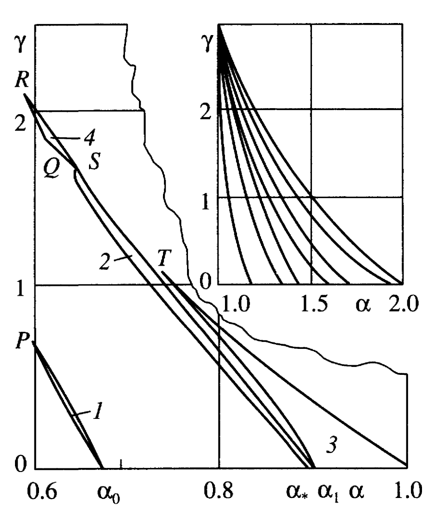

Fig. 5.

1 is enclosed between two curves, connecting the point (α_0, 0) and the point P (0.598, 0.665). For small γ these curves are given by the equations

$$\alpha_0^+ = \sigma_0(\gamma) + \frac{314867}{5038848}\gamma^4 + O(\gamma^5), \quad \alpha_0^- + \sigma_0(\gamma) + \frac{300755}{5038848}\gamma^4 + O(\gamma^5)$$

$$\sigma_0(\gamma) = \frac{2}{3} - \frac{2}{27}\gamma - \frac{17}{243}\gamma^3$$

Region 2 has the $\gamma = 0$ axis as its lower boundary of the section [α_*, α_1]. The equation of its left boundary, originating from the point (α_*, 0) for small γ has the form

$$\alpha_*(\gamma) = \frac{8}{9} - \frac{32}{243}\gamma - \frac{472}{6561}\gamma^2 + O(\gamma^3)$$

while the right boundary, originating from the point (α_0, 0), is given by the equation

$$\alpha_1^- = \alpha_1 - 0.1648\gamma + O(\gamma^2)$$

The right and left boundaries, as γ increases, intersect at the point S (0.632, 1.747), lying on the hyperbola $\gamma = -3 + 3/\alpha$, separating regions g_1 and g_2. Region 3 has the form of a curvilinear triangle. Its base is a section of the $\gamma = 0$ axis, connecting the points (α_1, 0) and (1, 0). The equation of the left side for small γ has the form

$$\alpha_1^+ = \alpha_1 - 0.1023\gamma + O(\gamma^2)$$

while the right side is a section of the hyperbola $\gamma = -3 + 3/\alpha$. The vertex of the triangle is the point T (0.729, 1.114).

For values of the parameters α and γ, lying in the region g_1 outside the regions 1, 2 and 3, the rotations of the satellite are unstable.

In region g_2, corresponding to rotation of the satellite around the axis of mean moment of inertia, there is one very narrow region of stability – the interior of the triangle QRS with vertices Q (0.614, 1.884), R (0.582, 2.175) and S (0.632, 1.747) (region 4 in Fig. 5). The base QS of the triangle is part of the hyperbola $\gamma = -3 + 3/\alpha$. Outside region 4 there is instability.

In the right upper part of Fig. 5 we show regions of stability and instability for the case of rotation around the axis of greatest moment of inertia (the region g_3 in Fig. 4). There are six regions of instability. They originate from the points (α_j, 0) ($j = 2, 3, \ldots, 7$) of the $\gamma = 0$ axis and for small γ are given by the following equations

$$\alpha_2^+ = \sigma_2(\gamma) + 0.1677\gamma^2 + O(\gamma^3), \quad \alpha_2^- = \sigma_2(\gamma) + 0.1648\gamma^2 + O(\gamma^3)$$

$$\sigma_2(\gamma) = \alpha_2 - 0.22689\gamma$$

$$\alpha_3^+ = \sigma_3(\gamma) + \frac{19}{162}\gamma^2 + O(\gamma^3), \quad \alpha_3^- = \sigma_3(\gamma) + \frac{55}{486}\gamma^2 + O(\gamma^3)$$

$$\sigma_3(\gamma) = \frac{4}{3} - \frac{8}{27}\gamma$$

$$\alpha_4^+ = \sigma_4(\gamma) - 0.054\gamma^3 + O(\gamma^4), \quad \alpha_4^- = \sigma_4(\gamma) - 0.06\gamma^3 + O(\gamma^4)$$

$$\sigma_4(\gamma) = \alpha_4 - 0.32026\gamma + 0.1603\gamma^2$$

$$\alpha_5^+ = \sigma_5(\gamma) - 0.055\gamma^3 + O(\gamma^4), \quad \alpha_5^- = \sigma_5(\gamma) - 0.057\gamma^3 + O(\gamma^4)$$

$$\sigma_5(\gamma) = \alpha_5 - 0.42696\gamma + 0.1612\gamma^2$$

$$\alpha_6^+ = \sigma_6(\gamma) - 0.0109\gamma^4 + O(\gamma^5), \quad \alpha_6^- = \sigma_6(\gamma) - 0.0112\gamma^4 + O(\gamma^5)$$

$$\sigma_6(\gamma) = \alpha_6 - 0.4707\gamma + 0.2045\gamma^2 - 0.078\gamma^3$$

$$\alpha_7^+ = \sigma_7(\gamma) - 0.063\gamma^5 + O(\gamma^6), \quad \alpha_7^- = \sigma_7(\gamma) - 0.064\gamma^5 + O(\gamma^6)$$

$$\sigma_7(\gamma) = \alpha_7 - 0.65097\gamma + 0.1281\gamma^2 - 0.014\gamma^3 + 0.1\gamma^4$$

The regions of instability are very narrow, and hence the representations of the boundaries of each of them on the right of the upper part of Fig. 5 merge. When $\gamma \to 3$ all the regions of instability tend asymptotically to the point (1, 3), corresponding to a satellite in the form of a thin rod ($A = C$, $B = 0$).

For values of the parameters α and γ from the region g_3, which do not belong to the six regions of instability indicated or their boundaries, the rotations of the satellite are stable.

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